

ISOTOPY CLASSIFICATION OF ENGEL STRUCTURES ON CIRCLE BUNDLES

MIRKO KLUKAS AND BIJAN SAHAMIE

ABSTRACT. We call two Engel structures isotopic if they are homotopic through Engel structures by a homotopy that fixes the characteristic line field. In the present paper we define an isotopy invariant of Engel structures on oriented circle bundles over closed oriented 3-manifolds and apply it to give an isotopy classification of Engel structures on circle bundles with characteristic line field tangent to the fibers.

1. INTRODUCTION

The present article deals with Engel structures on circle bundles over closed oriented 3-manifolds. For an introduction to basic notions of Engel structures we point the reader to [8, §2.2]. Our focus will lie on Engel structures with characteristic line field tangent to the fibers of the bundle. As mentioned in the abstract, we call two Engel structures **isotopic** if they are homotopic through Engel structures via a homotopy that fixes the characteristic line field. This notion is motivated by a result of Golubev who proved the existence of a version of Gray-stability for such homotopies (see [3] or [9, Theorem 3.50]). Let M be a closed oriented 3-manifold and $Q \rightarrow M$ an oriented circle bundle. For an Engel structure \mathcal{D} on Q with induced contact structure ξ on M we denote by $\phi_{\mathcal{D}}$ its associated development map (see §4). For the class of Engel structures we are considering, Engel structures and their development maps can be considered as equivalent objects. Development maps are fiberwise covering maps as observed in [6] and, therefore, to understand Engel structures on circle bundles we can equivalently try to understand fiberwise covering maps. To this end, in §3.1 we define a homotopy invariant of fiberwise covering maps we call the horizontal distance (see Definition 3.1). This distance – as the name suggests – measures *how far apart* two fiberwise coverings are, homotopically. In §3.2 we proceed with a homotopy classification of fiberwise covering maps by applying the horizontal distance (in arbitrary dimensions), where the homotopy goes through fiberwise covering maps. The 4-dimensional case of fiberwise coverings is then applied in §4 to the development maps of Engel structures: For Engel structures with characteristic line field tangent to the fibers there exists a natural analogue of the horizontal distance we denote by **twist** (see §4). Via the development maps, the invariant **twist** is identified with the horizontal distance. This allows us to apply the homotopy classification of fiberwise covering maps to prove the following Theorem 1.1. Here, we denote by $tw(\mathcal{D})$ the twisting number of an Engel structure \mathcal{D} .

Theorem 1.1. *Let Q be an oriented circle bundle over a closed oriented 3-manifold M . Two Engel structures \mathcal{D} and \mathcal{D}' with characteristic line field tangent to the fibers are isotopic*

if and only if $tw(\mathcal{D}) = tw(\mathcal{D}')$, their induced contact structures on the base agree and $\text{twist}(\mathcal{D}, \mathcal{D}') = 0$.

The theorem provides us with a set of invariants with which we can decide whether or not two given Engel structures are isotopic. To obtain a classification, we have to determine which sets of invariants can be realized by Engel structures. To this end, denote by $\text{Eng}(Q)$ the isotopy classes of Engel structures on Q with characteristic line field tangent to the fibers. For every contact structure ξ on M we denote by $\text{Eng}_\xi^n(Q) \subset \text{Eng}(Q)$ the isotopy classes of Engel structures with twisting number n and induced contact structure ξ on the base M . Similarly, we denote by $\text{Eng}_{\xi;o}^n(Q) \subset \text{Eng}_\xi^n(Q)$ the subset of isotopy classes of oriented Engel structures and $\text{Eng}_{\xi;no}^n(Q) \subset \text{Eng}_\xi^n(Q)$ the subset of non-orientable Engel structures.

Theorem 1.2. *Let Q be an oriented circle bundle over a closed oriented 3-manifold M .*

- (i) *For a number n and a contact structure ξ on M the set $\text{Eng}_\xi^n(Q)$ is non-empty if and only if $n \cdot e(Q) = 2 \cdot e(\xi)$.*
- (ii) *The set $\text{Eng}_{\xi;o}^n(Q)$ is non-empty if and only if n is even and $n/2 \cdot e(Q) = e(\xi)$.*
- (iii) *In case $\text{Eng}_\xi^n(Q)$ is non-empty, there is a simply-transitive $H^1(M; \mathbb{Z})$ -action on the set $\text{Eng}_\xi^n(Q)$. Furthermore, if $\text{Eng}_{\xi;o}^n(Q)$ is non-empty, the action on $\text{Eng}_\xi^n(Q)$ descends to a simply-transitive $2 \cdot H^1(M; \mathbb{Z})$ -action on $\text{Eng}_{\xi;o}^n(Q)$.*

A particularly nice special case of Theorem 1.2 are trivial circle bundles. In that case we obtain the following result.

Corollary 1.3. *The set $\text{Eng}(M \times \mathbb{S}^1)$ stays in one-to-one correspondence with elements in $\mathbb{Z} \times \Xi_2(M) \times H^1(M; \mathbb{Z})$ where $\Xi_2(M)$ denotes the set of contact structures on M with first chern class a 2-torsion class. Moreover, the one-to-one correspondence establishes a bijection from the isotopy classes of oriented Engel structures to the subset $2\mathbb{Z} \times \Xi_0(M) \times H^1(M; \mathbb{Z}) \subset \mathbb{Z} \times \Xi_2(M) \times H^1(M; \mathbb{Z})$ where $\Xi_0(M)$ denotes the set of contact structures on M with vanishing first chern class.*

The approach to the homotopical classification of fiberwise coverings chosen in this paper is very intuitive (see §3.1 and §3.2). As a small detour, we included §3.3 in which we discuss the relation of the present results with the results from [6]. In [6] we provided a classification of fiberwise coverings up to isomorphism of coverings in a very abstract way using spectral sequences and Čech cohomology. With our geometrically intuitive construction in the present article we are able to recover these *abstractly derived* results (see §3.3) using a more hands-on approach.

2. AN INTRODUCTORY EXAMPLE

In this section we briefly discuss an introductory example without going into too many details. The purpose is to indicate the various ways our homotopical invariant of fiberwise coverings can be interpreted and formulated.

2.1. The Covering-Space Theoretic Viewpoint. The 2-dimensional torus T^2 is a circle bundle over \mathbb{S}^1 . Obviously, the map $\phi_0(\mathbf{p}, \theta) = (\mathbf{p}, n\theta)$ is a fiberwise n -fold covering. Furthermore, for an integer $\alpha \in \mathbb{Z}$ we can define a different fiberwise n -fold covering by $\phi_\alpha(\mathbf{p}, \theta) = (\mathbf{p}, n \cdot \theta + \alpha \cdot \mathbf{p})$. Up to homotopy, these are all existing fiberwise n -fold coverings of the circle bundle T^2 by the homotopy invariance of $(\phi_\alpha)_*$ on the first homology $H_1(T^2; \mathbb{Z})$.

Going up two dimensions, the 4-torus T^4 is a circle bundle over T^3 . Suppose we are given fiberwise covering maps $\phi_i: T^4 \rightarrow T^4$, $i = 1, 2$, and a loop $\gamma: \mathbb{S}^1 \rightarrow T^3$. On the pullback γ^*T^4 the map ϕ_i induces a fiberwise covering

$$\widetilde{\phi}_i: \gamma^*T^4 \rightarrow \gamma^*T^4,$$

whose homotopy class is uniquely determined by the homotopy class of ϕ_i . The circle bundle γ^*T^4 is trivial. After choosing a trivialization, $\widetilde{\phi}_i$ corresponds to some $\phi_{\alpha_\gamma^i}$ for a suitable $\alpha_\gamma^i \in \mathbb{Z}$. It is plausible to expect that α_γ^i is invariant under homotopies of ϕ_i and that it just depends on the homology class of the loop γ in T^3 . Let us accept this for now (see §2.2, §2.3 and the proof of Theorem 3.2). Although the assignment $\gamma \mapsto \alpha_\gamma^i$ depends on the trivialization of γ^*T^4 , the difference $\alpha_\gamma^1 - \alpha_\gamma^2$ is certainly independent of it. If we choose a trivialization $\gamma^*T^4 \cong T^2$, then $\widetilde{\phi}_1$ and $\widetilde{\phi}_2$ correspond to maps which can be written as matrices

$$\widetilde{\phi}_1 = \begin{pmatrix} 1 & 0 \\ \alpha_\gamma^1 & n \end{pmatrix} \quad \text{and} \quad \widetilde{\phi}_2 = \begin{pmatrix} 1 & 0 \\ \alpha_\gamma^2 & n \end{pmatrix}.$$

From this we can read off the equality

$$(2.1) \quad \begin{aligned} (\alpha_\gamma^1 - \alpha_\gamma^2)[F] &= (p_F)_*((\phi_1)_* - (\phi_2)_*)[\gamma] \\ &= [p_F \circ \phi_1 \circ \gamma] - [p_F \circ \phi_2 \circ \gamma], \end{aligned}$$

where $[F]$ is the fundamental class of a fiber and $p_F: T^4 \rightarrow \mathbb{S}^1$ the projection onto the fiber.

2.2. The Interpretation as Twists. An alternative viewpoint can be derived easily from the given discussion. As mentioned above, the α_γ^i 's just depend on the homotopy type of ϕ_i and the homology class of the γ 's. Hence, the assignment $\gamma \mapsto \alpha_\gamma^1 - \alpha_\gamma^2$ can be reduced to a map

$$(2.2) \quad d(\phi_1, \phi_2): H_1(T^3; \mathbb{Z}) \rightarrow \mathbb{Z}, [\gamma] \mapsto \alpha_\gamma^1 - \alpha_\gamma^2,$$

which carries the relevant homotopical information. So,

$$(2.3) \quad \phi_\alpha(\mathbf{p}, \theta) = (\mathbf{p}, n\theta + \langle \alpha, \mathbf{p} \rangle),$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}^3$, all represent different homotopy classes of fiberwise n -fold covering maps. Comparing with the discussion from §2.1, we see that the ϕ_α have to be distinct, pairwise. Namely, for a standard basis $[\gamma_i]$, $i = 1, 2, 3$, of $H_1(T^3; \mathbb{Z})$ with $\phi_1 = \phi_\alpha$ and $\phi_2 = \phi_0$ we compute

$$d(\phi_\alpha, \phi_0)[F] = (\alpha_{\gamma_i}^1 - \alpha_{\gamma_i}^2)[F] = \alpha_i[F],$$

where the first equality is given by definition of $d(\phi_\alpha, \phi_0)$ and the second equality is given by (2.1). So, $d(\phi_\alpha, \phi_0)$ is the morphism on $H_1(T^3; \mathbb{Z})$ which sends $[\gamma_i]$ to α_i for $i = 1, 2, 3$. Moreover, the right hand side of the Equation (2.1) we used here determines the number

of times ϕ_1 moves around the fiber relative to ϕ_2 as we move along γ . We see by (2.2) and Equation (2.1) that this is precisely what is measured by $d(\phi_\alpha, \phi_0)$.

2.3. The Group Action. Another way of formulating the definition of ϕ_α is

$$\phi_\alpha(\mathbf{p}, \theta) = (\mathbf{p}, n\theta + \langle \alpha, \mathbf{p} \rangle) = \langle \alpha, \mathbf{p} \rangle \cdot (\mathbf{p}, n\theta) = \langle \alpha, \mathbf{p} \rangle \cdot \phi_0(\mathbf{p}, \theta),$$

where the second equality uses the \mathbb{S}^1 -action on the fibers of T^4 . If we define a map $f: T^3 \rightarrow \mathbb{S}^1$ by $f(\mathbf{p}) = \langle \alpha, \mathbf{p} \rangle$, then $\phi_\alpha(\mathbf{p}, \theta) = f(\mathbf{p}) \cdot \phi_0(\mathbf{p}, \theta)$. In fact, every map ϕ_α can be obtained from ϕ_0 by using a suitable map $T^3 \rightarrow \mathbb{S}^1$ and the group action as above. It is not hard to see that a homotopy of ϕ_α through fiberwise covering maps is equivalent to a homotopy of the map f . The homotopy type of f is captured by the class $f^*[\mathbb{S}^1] \in H^1(T^3; \mathbb{Z})$ where $[\mathbb{S}^1]$ is a fundamental class of \mathbb{S}^1 . By the universal coefficient theorem, $H^1(T^3; \mathbb{Z})$ is isomorphic to $\text{Hom}(H_1(T^3; \mathbb{Z}); \mathbb{Z})$. Therefore, the class $f^*[\mathbb{S}^1]$ is uniquely determined by evaluation on a basis of $H_1(T^3; \mathbb{Z})$. Using the standard basis $[\gamma_i]$, $i = 1, 2, 3$, of the first homology of T^3 , we obtain

$$(f^*[\mathbb{S}^1])[\gamma_i] = \alpha_i$$

for $i = 1, 2, 3$. More generally, given two fiberwise coverings ϕ_i , $i = 1, 2$, and let $f: T^3 \rightarrow \mathbb{S}^1$ be given such that $\phi_2 = f \cdot \phi_1$, then for every loop γ we have

$$\alpha_\gamma^1 - \alpha_\gamma^2 = (f^*[\mathbb{S}^1])[\gamma].$$

So, in particular, $d(\phi_1, \phi_2)[\gamma] = (f^*[\mathbb{S}^1])[\gamma]$.

2.4. Relation to Engel Structures on the Four-Dimensional Torus. It is easily possible to define Engel structures on the four-dimensional torus. Let ξ be the contact structure on the base space which is given as the kernel of the 1-form $\sin(2\pi z)dx + \cos(2\pi z)dy$. The contact planes are spanned by the vector field ∂_z and

$$V_{\mathbf{p}} = \cos(2\pi z)\partial_x + \sin(2\pi z)\partial_y$$

where \mathbf{p} is a point in T^3 . On T^4 , with coordinates (\mathbf{p}, θ) , we define an Engel structure $\mathcal{D}_\alpha^n(\xi)$ as the 2-planes spanned by ∂_θ and

$$\cos(\pi(n\theta + \langle \alpha, \mathbf{p} \rangle))\partial_z + \sin(\pi(n\theta + \langle \alpha, \mathbf{p} \rangle))V_{\mathbf{p}}.$$

There is a natural map $\phi: T^4 \rightarrow T^4$ called the development map. The development map of this Engel structure is the fiberwise covering map ϕ_α . We should see that the data of the Engel structure translate into the data that determine the homotopy type of ϕ_α : Pick the standard basis $[\gamma_i]$, $i = 1, 2, 3$, of $H_1(M; \mathbb{Z})$ represented by the obvious loops γ_i , then $\mathcal{D}_\alpha^n(\xi)|_{\gamma_i}$ determines a family of 1-dimensional subspaces in $\xi|_{\gamma_i}$. What we see is that by the summand $\langle \alpha, \cdot \rangle$ the subspaces of $\xi|_{\gamma_i}$ associated to $\mathcal{D}_\alpha^n(\xi)|_{\gamma_i}$ make α_i -full turns inside $\xi|_{\gamma_i}$ relative to the family of subspaces given by $\mathcal{D}_0^n(\xi)|_{\gamma_i}$.

3. HOMOTOPICAL CLASSIFICATION OF FIBERWISE COVERINGS

For this section suppose we are given a closed oriented manifold M and two circle bundles Q and P over M . Recall that a map $\phi: Q \rightarrow P$ is called a fiberwise n -fold covering map if its restriction to every fiber Q_p , $p \in M$, equals the standard n -fold covering map $\varphi_n: \mathbb{S}^1 \rightarrow \mathbb{S}^1$, $\theta \mapsto n \cdot \theta$. To be more precise, recall that there are simply-transitive \mathbb{S}^1 -actions on the fibers of Q and P . Fixing an element in the fiber Q_p and P_p , the group actions provide us with an identification of Q_p and P_p with \mathbb{S}^1 . Under these identifications, the map $\phi|_{Q_p}$ corresponds to φ_n .

Remark 1. A priori, one could define fiberwise n -fold coverings by requiring the restriction $\phi|_{Q_p}$ to be an n -fold covering but not specifically φ_n . However, these two notions are equivalent up to homotopy. In fact, a fiberwise covering of the *more general form* can be homotoped into a fiberwise covering of the *restricted type* we are considering in this article.

3.1. The Horizontal Distance. Let G be an abelian group, n a natural number and denote by $K(G, n)$ the associated Eilenberg-MacLane space. Recall that for every CW-complex X there is a natural bijection

$$[X; K(G, n)] \rightarrow H^n(X; \mathbb{Z}), [f] \mapsto f^*(\alpha_0),$$

where $\alpha_0 \in H^n(K(G, n); G)$ is a fixed fundamental class of $K(G, n)$ (see [5, Theorem 4.57]). We will apply this classification to fiberwise coverings: Namely, given two n -fold fiberwise covering maps $\phi_i: Q \rightarrow P$, $i = 1, 2$, for every $p \in M$ we define $f_{(\phi_1, \phi_2)}(p) \in \mathbb{S}^1$ as the element defined via the equation

$$f_{(\phi_1, \phi_2)}(p) \cdot \phi_1(q) = \phi_2(q),$$

where q is an arbitrary point in Q_p . This definition does not depend on the point q . If q' is another point in the fiber over p , then there exists an element $\theta \in \mathbb{S}^1$ such that $\theta \cdot q = q'$. By the chain of equalities

$$\begin{aligned} \phi_1(q') &= \phi_1(\theta \cdot q) = (n \cdot \theta) \cdot \phi_1(q) = (n \cdot \theta + f_{(\phi_1, \phi_2)}(p)) \cdot \phi_2(q') \\ &= (f_{(\phi_1, \phi_2)}(p) + n \cdot \theta) \cdot \phi_2(q) = f_{(\phi_1, \phi_2)}(p) \cdot \phi_2(\theta \cdot q) \\ &= f_{(\phi_1, \phi_2)}(p) \cdot \phi_2(q') \end{aligned}$$

well-definedness follows. Here, we applied the assumption that ϕ_1 and ϕ_2 are both fiberwise covering maps. Thus, we are provided with a map

$$f_{(\phi_1, \phi_2)}: M \rightarrow \mathbb{S}^1.$$

We will prove that the homotopy type of this map *measures the homotopical distance* between ϕ_1 and ϕ_2 . To this end, we need a homotopical classification of maps into \mathbb{S}^1 . Note that \mathbb{S}^1 equals $K(\mathbb{Z}, 1)$ and, therefore, we have a bijection

$$[M; \mathbb{S}^1] \rightarrow H^1(M; \mathbb{Z}), [f] \mapsto f^*([\mathbb{S}^1]),$$

where $[\mathbb{S}^1]$ is the fundamental class of $[\mathbb{S}^1]$ that represents the natural orientation on the \mathbb{S}^1 -fibers.

Definition 3.1. Given two fiberwise n -fold coverings $\phi_i: Q \rightarrow P$, $i = 1, 2$, we denote by $d(\phi_1, \phi_2)$ the class $f_{(\phi_1, \phi_2)}^*([\mathbb{S}^1]) \in H^1(M; \mathbb{Z})$ and call it **the horizontal distance** between ϕ_1 and ϕ_2 .

The example of the 4-torus discussed in §2 shows that different fiberwise coverings can be constructed by adding twists along loops in the base space (see §2.2). More precisely, we have seen in the specific case of the 4-torus that the horizontal distance between two fiberwise coverings ϕ_1 and ϕ_2 measures some sort of twisting of ϕ_1 relative to ϕ_2 along loops in the base space (see §2.1 and §2.2). What is done for the 4-torus can be done in the general setting as well. Moreover, observe that $d(\phi_1, \phi_2) \frown [\gamma] = d(\phi_1, \phi_2)([\gamma])$ where on the right we interpret the class $d(\phi_1, \phi_2)$ as an element of $\text{Hom}(H_1(M; \mathbb{Z}), \mathbb{Z}) \cong H^1(M; \mathbb{Z})$ by the universal coefficient theorem. However, we have

$$d(\phi_1, \phi_2)[\gamma] = f_{(\phi_1, \phi_2)}^*([\mathbb{S}^1])[\gamma] = [f_{(\phi_1, \phi_2)} \circ \gamma]$$

and the homotopy class of $f_{(\phi_1, \phi_2)} \circ \gamma: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ precisely measures the relative twisting discussed in §2.2.

3.2. Homotopy Classification. Let Q and P be two circle bundles over a closed oriented manifold. Denote by $\text{Cov}(Q, P)$ the homotopy classes of fiberwise covering maps where homotopies move through fiberwise coverings and for every $n \in \mathbb{N}$ denote by $\text{Cov}_n(Q, P)$ the subset consisting of homotopy classes of n -fold coverings. Our goal is to prove the following theorem.

Theorem 3.2. *For a number $n \in \mathbb{N}$, the set $\text{Cov}_n(Q, P)$ is non-empty if and only if $n \cdot e(Q) = e(P)$. If it is non-empty, there is a simply-transitive group action of $H^1(M; \mathbb{Z})$ on $\text{Cov}_n(Q, P)$. Hence, there is a one-to-one correspondence between $\text{Cov}_n(Q, P)$ and $H^1(M; \mathbb{Z})$.*

In order to prove this result, we will need the following two preparatory lemmas.

Lemma 3.3. *Suppose we are given two fiberwise covering maps $\phi_i: Q \rightarrow P$, $i = 1, 2$. These maps are homotopic through fiberwise covering maps if and only if they have the same number of sheets and their horizontal distance vanishes.*

Proof. If the maps are homotopic as fiberwise covering maps, then the map $f_{(\phi_1, \phi_2)}$ is homotopic to a constant map. Hence, their horizontal distance vanishes.

Conversely, suppose that ϕ_i , $i = 1, 2$, have the same number of sheets and have vanishing horizontal distance. We define a homotopy

$$F: Q \rightarrow [0, 1] \rightarrow Q, (q, t) \mapsto h(\pi(q), t) \cdot \phi_1(q),$$

where $h: M \times [0, 1] \rightarrow \mathbb{S}^1$ is a homotopy from $f_{(\phi_1, \phi_2)}$ to the constant map c_1 which maps every point to 1. \square

Lemma 3.4. *Given fiberwise covering maps $\phi_i: Q \rightarrow P$, $i = 1, 2, 3$, of a circle bundle $P \rightarrow M$ over a closed oriented manifold M we have $d(\phi_1, \phi_2) + d(\phi_2, \phi_3) = d(\phi_1, \phi_3)$.*

Proof. Given a point $q \in Q$ we have $f_{(\phi_1, \phi_3)}(q) \cdot \phi_1(q) = \phi_3(q)$. Furthermore, $f_{(\phi_1, \phi_3)}(q)$ is uniquely determined by this equality. However,

$$\begin{aligned} (f_{(\phi_1, \phi_3)}(q) + f_{(\phi_1, \phi_2)}(q)) \cdot \phi_1(q) &= f_{(\phi_2, \phi_3)}(q) \cdot (f_{(\phi_1, \phi_2)}(q) \cdot \phi_1(q)) = f_{(\phi_2, \phi_3)}(q) \cdot \phi_2(q) \\ &= \phi_3(q) \\ &= f_{(\phi_1, \phi_3)}(q) \cdot \phi_1(q). \end{aligned}$$

Since the action on the fibers is simply-transitive, the sum $f_{(\phi_2, \phi_3)}(q) + f_{(\phi_1, \phi_2)}(q)$ equals $f_{(\phi_1, \phi_3)}(q)$. Hence,

$$d(\phi_1, \phi_3) = f_{(\phi_1, \phi_3)}^*[\mathbb{S}^1] = f_{(\phi_2, \phi_3)}^*[\mathbb{S}^1] + f_{(\phi_1, \phi_2)}^*[\mathbb{S}^1] = d(\phi_1, \phi_2) + d(\phi_2, \phi_3),$$

which finishes the proof. \square

Proof of Theorem 3.2. Given a fiberwise covering $\phi_1: Q \rightarrow P$ and a class $\alpha \in H^1(M; \mathbb{Z})$, denote by $f: M \rightarrow \mathbb{S}^1$ a map with $f^*[\mathbb{S}^1] = \alpha$. Then, define

$$(3.1) \quad f \cdot \phi_1: Q \rightarrow P, \quad q \mapsto f(\pi(q)) \cdot \phi_1(q),$$

where $\pi: Q \rightarrow M$ is the canonical projection map. We define $\alpha \cdot [\phi_1] := [f \cdot \phi_1]$. The homotopy class of $f \cdot \phi_1$ does not depend on the specific choice of f which can be derived directly using Lemma 3.4. So, we are provided with a map

$$\kappa: H^1(M; \mathbb{Z}) \rightarrow \text{Cov}_n(Q, P), \quad \alpha \mapsto \alpha \cdot [\phi_1].$$

By construction, $d(\phi_1, \kappa(\alpha)) = \alpha$. So, the last two lemmas show that this is a bijection: Namely, given α, α' such that $\kappa(\alpha) = \kappa(\alpha')$ then

$$\alpha - \alpha' = d(\phi_1, \kappa(\alpha)) + d(\kappa(\alpha'), \phi_1) = d(\phi_1, \phi_1) = 0$$

by Lemma 3.4. Hence, $\alpha = \alpha'$ which shows injectivity.

To prove surjectivity, pick a fiberwise n -fold covering $\phi_2: Q \rightarrow P$ and denote by α the class $d(\phi_1, \phi_2)$. Now we have

$$d(\phi_2, \kappa(\alpha)) = d(\phi_2, \phi_1) + d(\phi_1, \kappa(\alpha)) = 0$$

by Lemma 3.4. Finally, Lemma 3.3 shows that the fiberwise coverings ϕ_2 and $\kappa(\alpha)$ are homotopic through fiberwise covering maps.

To prove the existence statement in the theorem, assume that $n \cdot e(Q) = e(P)$. Denote by $\Sigma \subset M$ a submanifold which represents the homology class $\text{PD}[e(Q)] \in H_{n-2}(M; \mathbb{Z})$ and denote by $\nu\Sigma$ its normal bundle. Since $U(1)$ is a retract of $\text{Diff}(\mathbb{S}^1)$, without loss of generality we may assume that Q determines a two-dimensional real vector bundle $E_Q \rightarrow M$ for which Q is the unit-sphere bundle after choosing a suitable metric g . The homology class of Σ is the homology class of the zero locus of a section $s: M \rightarrow E_Q$ which intersects the zero section of E_Q transversely. So, we may assume that $s^{-1}(0) = \Sigma$. Consequently, by normalizing s , we obtain a section of $Q|_{M \setminus \nu\Sigma}$. Therefore, we can write Q as

$$(3.2) \quad Q = (M \setminus \nu\Sigma) \times \mathbb{S}^1 \cup_{\psi} (\nu\Sigma \times \mathbb{S}^1)$$

where $\psi = (\psi_1, \psi_2): \partial\nu\Sigma \times \mathbb{S}^1 \longrightarrow \partial(M \setminus \nu\Sigma) \times \mathbb{S}^1$ denotes a suitable gluing map. We define a circle bundle P_η by

$$(3.3) \quad P_\eta = (M \setminus \nu\Sigma) \times \mathbb{S}^1 \cup_\eta (\nu\Sigma \times \mathbb{S}^1)$$

where η is the gluing map (ψ_1, ψ_2^n) . The Euler class of P_η is $n \cdot e(Q)$ which equals the Euler class of P . So, P_η is isomorphic to P and we may use P_η as a model for P . Now, there is an obvious fiberwise n -fold covering map from Q to P , namely the map

$$\phi: Q \longrightarrow P, [(p, \theta)] \longmapsto [(p, n \cdot \theta)].$$

Here, we used the description (3.2) of Q and the description (3.3) of P .

Conversely, assuming that there is a fiberwise n -fold covering $\phi: Q \longrightarrow P$, we have to prove that $n \cdot e(Q) = e(P)$. We either use a similar reasoning as above, or we apply [6, Lemma 3.1]. \square

3.3. Relations to Previous Results. The horizontal distance also detects if two fiberwise covering maps $\phi_i: Q \longrightarrow P$, $i = 1, 2$, are isomorphic as fiberwise covering maps.

Proposition 3.5. *Two fiberwise coverings $\phi_i: Q \longrightarrow P$, $i = 1, 2$, are isomorphic as fiberwise coverings if and only if they have the same number n of sheets and $d(\phi_1, \phi_2) \bmod n = 0$.*

Proof. Suppose that ϕ_i , $i = 1, 2$, are isomorphic as fiberwise n -fold covering maps and denote by $\psi: Q \longrightarrow Q$ an isomorphism. It is easy to see that ψ also commutes with the bundle projection $Q \longrightarrow M$. Hence, ψ is also an automorphism of the circle bundle $Q \longrightarrow M$. Every such bundle isomorphism can be characterized by a map $\psi_2: M \longrightarrow \mathbb{S}^1$ by the equation

$$\psi(q) = \psi_2(\pi(q)) \cdot q.$$

Note that ψ_2 is well-defined since ψ maps fibers to fibers. We have to compute $d(\phi_1, \phi_2) = d(\phi_1, \phi_1 \circ \psi)$. For a point $q \in Q$ consider

$$f_{(\phi_1, \phi_1 \circ \psi)}(q) \cdot \phi_1(q) = \phi_1(\psi(q)) = \phi_1(\psi_2(q) \cdot q) = (n \cdot \psi_2(q)) \cdot \phi_1(q),$$

which implies that $f_{(\phi_1, \phi_1 \circ \psi)}(q) = n \cdot \psi_2(q)$. Consequently, given a loop γ in Q then $(f_{(\phi_1, \phi_1 \circ \psi)})_*([\gamma]) = n \cdot (\psi_2)_*[\gamma]$. Considering the diagram

$$\begin{array}{ccccccc} H^1(\mathbb{S}^1; \mathbb{Z}) & \xrightarrow{\cong} & \text{Hom}(H_1(\mathbb{S}^1; \mathbb{Z}), \mathbb{Z}) & \xlongequal{\quad} & \text{Hom}(H_1(\mathbb{S}^1; \mathbb{Z}), \mathbb{Z}) & \xleftarrow{\cong} & H^1(\mathbb{S}^1; \mathbb{Z}) \\ \downarrow f_{(\phi_1, \phi_1 \circ \psi_2)}^* & & \downarrow ((f_{(\phi_1, \phi_1 \circ \psi)})_*)^* & & \downarrow (n \cdot (\psi_2)_*)^* & & \downarrow n \cdot \psi_2^* \\ H^1(Y; \mathbb{Z}) & \xrightarrow{\cong} & \text{Hom}(H_1(Y; \mathbb{Z}), \mathbb{Z}) & \xlongequal{\quad} & \text{Hom}(H_1(Y; \mathbb{Z}), \mathbb{Z}) & \xleftarrow{\cong} & H^1(Y; \mathbb{Z}) \end{array}$$

our discussion shows that the square in the middle commutes. By the universal coefficient theorem the left and the right square of the diagram commute. Thus,

$$d(\phi_1, \phi_1 \circ \psi_1) = n \cdot \psi_2^*[\mathbb{S}^1]$$

which makes it vanish modulo- n .

Conversely, given two fiberwise n -fold coverings $\phi_i: Q \longrightarrow P$, $i = 1, 2$, such that $d(\phi_1, \phi_2)$ vanishes modulo n this means that $d(\phi_1, \phi_2) = n \cdot \alpha$ for some suitable class $\alpha \in H^1(M; \mathbb{Z})$. Denote by $\psi_2: M \longrightarrow \mathbb{S}^1$ a continuous map for which $\psi_2^*[\mathbb{S}^1] = \alpha$. We construct an

associated automorphism ψ of the bundle $Q \rightarrow M$ by sending an element $q \in Q$ to $\psi(q) = \psi_2(\pi(q)) \cdot q$. By the considerations from above, we get $d(\phi_1, \phi_1 \circ \psi) = n \cdot \alpha$. Consequently, we have

$$d(\phi_1 \circ \psi, \phi_2) = d(\phi_1 \circ \psi, \phi_1) + d(\phi_1, \phi_2) = -n \cdot \alpha + n \cdot \alpha = 0,$$

where the first and second equality holds by Lemma 3.4. Theorem 3.2 implies that $\phi_1 \circ \psi$ and ϕ_2 are homotopic. So, after possibly defining ψ (resp. ψ_2) differently, $\phi_1 \circ \psi$ equals ϕ_2 which finishes the proof. \square

Remark 2. The last step in the proof of Proposition 3.5 used the fact that a homotopy of $\phi_1 \circ \psi$ through fiberwise covering maps amounts to a homotopy of ψ_2 . We leave the proof of this fact to the interested reader.

In [6] we already identified a cohomology class which provides a classification of fiberwise covering maps up to isomorphism of coverings. However, the definition of the class provided there was on a very abstract level using spectral sequences and Čech cohomology. The following result shows that our geometrically defined class $d(\phi_1, \phi_2)$ reduced modulo n equals the class defined in [6] (see [6, Theorem 3.3] and cf. [6, Corollary 4.1]).

Proposition 3.6. *Given two fiberwise n -fold coverings $\phi_i: Q \rightarrow P$, $i = 1, 2$, and let $\alpha_{\phi_i} \in H^1(M; \mathbb{Z}_n)$, $i = 1, 2$, be their associated classes defined via Čech cohomology using the approach in [6]. The classes $\alpha_{\phi_1} - \alpha_{\phi_2}$ and $d(\phi_1, \phi_2) \bmod n$ are equal.*

Sketch of the Proof. In [6, Theorem 1.1] we gave a characterization of fiberwise n -fold coverings via the following exact sequence.

$$0 \longrightarrow H^1(M; \mathbb{Z}_n) \xrightarrow{\pi^*} H^1(P; \mathbb{Z}_n) \xrightarrow{\iota^*} H^1(\mathbb{S}^1; \mathbb{Z}_n) \xrightarrow{d} H^2(M; \mathbb{Z}_n).$$

We have shown that fiberwise coverings correspond to classes $\beta \in H^1(P; \mathbb{Z}_n)$ for which $\iota_*(\beta) = [\varphi_n]$ where φ_n is the unique (up to homotopy) connected n -fold covering of \mathbb{S}^1 . Thus, there is a simply transitive group action of $H^1(M; \mathbb{Z}_n)$ on the set of n -fold fiberwise coverings given via

$$\begin{array}{ccc} H^1(M; \mathbb{Z}_n) & \times & (\iota_*)^{-1}([\varphi_n]) \\ (\alpha & , & \beta) \end{array} \longrightarrow \begin{array}{ccc} & & (\iota_*)^{-1}([\varphi_n]) \\ & \longmapsto & \beta + \pi^*(\alpha). \end{array}$$

Choose an open covering \mathcal{U} of M and let us denote by $N(\mathcal{U})$ the nerve of the covering. Without loss of generality the covering \mathcal{U} can be chosen fine enough such that $\pi^*(\alpha)$ is the image of a suitable class, $\alpha_{\mathcal{U}}$ say, in

$$H^1(N(\mathcal{U}); \mathbb{Z}) \longrightarrow \varinjlim H^1(N(\mathcal{V}), \mathbb{Z}) = \varinjlim \text{Hom}(H_1(N(\mathcal{V}); \mathbb{Z}), \mathbb{Z}),$$

where the last equality is provided by the universal coefficient theorem and the naturality of the universal coefficient theorem exact sequence. Therefore, the class $\alpha_{\mathcal{U}}$ can be seen as a morphism on the first homology of the nerve and, as such, it is characterized by its image on a basis of $H_1(N(\mathcal{U}); \mathbb{Z})$. With a little effort one can see that $\alpha_{\mathcal{U}}$ on $H_1(N(\mathcal{U}); \mathbb{Z})$ has an analogous description as the horizontal distance on $H^1(M; \mathbb{Z})$. Furthermore, it is not hard to see that this description is preserved by the direct limit, which shows that $\alpha_{\mathcal{U}}$ converges to a class whose definition coincides with the definition of the horizontal distance. \square

4. ENGEL STRUCTURES ON CIRCLE BUNDLES

Given a circle bundle $Q \rightarrow M$ over a closed oriented 3-manifold M with Engel structure \mathcal{D} whose characteristic line field is tangent to the fibers, we associate to it the so-called development map

$$\phi_{\mathcal{D}}: (Q, \mathcal{D}) \rightarrow (\mathbb{P}\xi, \mathcal{D}\xi)$$

where $(\mathbb{P}\xi, \mathcal{D}\xi)$ is the prolongation of the contact structure ξ induced on the base space M (see [6, §4]). We have seen in [6] that $\phi_{\mathcal{D}}$ is a fiberwise covering map where the twisting number $tw(\mathcal{D})$ equals the number of sheets of this covering. An isotopy of Engel structures, i.e. a homotopy through Engel structures whose characteristic line field is tangent to the fibers, leaves fixed the induced contact structure on the base. Hence, an isotopy of the Engel structure \mathcal{D} is equivalent to a homotopy of $\phi_{\mathcal{D}}$ through fiberwise covering maps (see Theorem 1.1). As a result, §3.2 provides us with a method to classify such Engel structures up to isotopy.

Given another Engel structure \mathcal{D}' then to every loop γ in M we can associate an integer in the following way: Lift the loop γ to a loop $\tilde{\gamma}$ in Q . The Engel structures \mathcal{D} and \mathcal{D}' induce \mathbb{S}^1 -families of 1-dimensional subspaces of the trivial bundle $\xi|_{\pi(\tilde{\gamma})}$. These paths of subspaces can be interpreted as sections of $\mathbb{P}\xi|_{\tilde{\gamma}}$. If we choose the path coming from \mathcal{D} as a reference, it provides us with an identification $\mathbb{P}\xi|_{\tilde{\gamma}} \cong \gamma \times \mathbb{S}^1$. With this identification, to the path coming from \mathcal{D}' we can assign an integer by measuring the number times the path associated to \mathcal{D}' moves through the \mathbb{S}^1 -fiber of $\mathbb{P}\xi|_{\tilde{\gamma}}$. This number shall be denoted by $\mathbf{twist}(\mathcal{D}, \mathcal{D}')(\gamma)$. In fact, we see that

$$(4.1) \quad \mathbf{twist}(\mathcal{D}, \mathcal{D}')(\gamma) = d(\phi_{\mathcal{D}}, \phi_{\mathcal{D}'})_*[\gamma].$$

Thus, the given assignment descends to a map

$$\mathbf{twist}(\mathcal{D}, \mathcal{D}'): H_1(M; \mathbb{Z}) \rightarrow \mathbb{Z}.$$

The invariant \mathbf{twist} is the invariant coming from the intuition that the Engel structure on a circle bundle with characteristic line field tangent to the fibers is basically given by the contact structure on a cross section, its twisting along fibers (vertical twisting), and its twisting along non-trivial loops in the base space (horizontal twisting, cf. §2.4). The Equality (4.1) indicates that this is an isotopy invariant and that it should be strong enough to give a classification.

Proof of Theorem 1.1. Given two Engel structures \mathcal{D}_0 and \mathcal{D}_1 which are isotopic and denote by $(\mathcal{D}_t)_{t \in [0,1]}$ a corresponding homotopy through Engel structures. There exists an isotopy $\psi_t: Q \rightarrow Q$ such that $(\psi_t)_*\mathcal{D}_0 = \mathcal{D}_t$ (see [3] or [9, Theorem 3.50]). Since the homotopy fixes the characteristic line field, the isotopy ψ_t is an automorphism of the circle bundle $Q \rightarrow M$. Denote by \mathcal{E}_t the associated even contact structure of \mathcal{D}_t . Since $\mathcal{E}_t = [\mathcal{D}_t, \mathcal{D}_t]$ by definition, we have

$$\mathcal{E}_t = [\mathcal{D}_t, \mathcal{D}_t] = [(\psi_t)_*(\mathcal{D}_0), (\psi_t)_*(\mathcal{D}_0)] = (\psi_t)_*[\mathcal{D}_0, \mathcal{D}_0] = (\psi_t)_*\mathcal{E}_0.$$

Thus,

$$\xi_t = \pi_*(\psi_t)_*(\mathcal{E}_0) = (\pi \circ \psi_t)_*\mathcal{E}_0 = \pi_*\mathcal{E}_0 = \xi_0.$$

So, the homotopy of Engel structures apparently fixes the contact structure on the base space. Therefore, the triangle

$$\begin{array}{ccc} Q & \xrightarrow{\psi_t} & Q \\ & \searrow \phi_{\mathcal{D}_0} & \swarrow \phi_{\mathcal{D}_t} \\ & \mathbb{P}\xi_0 & \end{array}$$

commutes for every $t \in [0, 1]$, which shows that the development maps $\phi_{\mathcal{D}_i}$, $i = 0, 1$, are homotopic through fiberwise covering maps. Thus, $\text{twist}(\mathcal{D}_0, \mathcal{D}_1) = d(\phi_{\mathcal{D}_0}, \phi_{\mathcal{D}_1}) = 0$.

Conversely, if $\text{twist}(\mathcal{D}_0, \mathcal{D}_1) = d(\phi_{\mathcal{D}_0}, \phi_{\mathcal{D}_1}) = 0$, the two development maps $\phi_{\mathcal{D}_0}$ and $\phi_{\mathcal{D}_1}$ are homotopic through fiberwise covering maps. Denote by $(\phi_t)_{t \in [0, 1]}$ the associated homotopy and denote by $\mathcal{D}\xi_0$ the prolonged Engel structure on $\mathbb{P}\xi_0$. For every $t \in [0, 1]$ there exists a unique distribution of 2-planes \mathcal{D}_t on Q such that $(\phi_t)_*\mathcal{D}_t = \mathcal{D}\xi_0$. The family of 2-planes $(\mathcal{D}_t)_{t \in [0, 1]}$ defines a homotopy of Engel structures from \mathcal{D}_0 to \mathcal{D}_1 . \square

An Engel structure \mathcal{D} on Q with characteristic line field tangent to the fibers induces a contact structure ξ on the base space M . To the contact structure we can associate its prolongation $(\mathbb{P}\xi, \mathcal{D}\xi)$. As noted at the beginning of this section, to the Engel structure \mathcal{D} we can associate the development map $\phi_{\mathcal{D}}: Q \rightarrow \mathbb{P}\xi$ which is a fiberwise $tw(\mathcal{D})$ -fold covering. An isotopy of the Engel structure \mathcal{D} , i.e. a homotopy through Engel structures which fixes the characteristic line field, is equivalent to a homotopy of the development map through fiberwise covering maps (see Theorem 1.1). Hence, the existence and classification of Engel structures up to isotopy – where we consider Engel structures with characteristic line field tangent to the fibers – is equivalent to the existence and classification of fiberwise covering maps $Q \rightarrow P$ up to homotopy through fiberwise coverings. In case of oriented Engel structures this approach has a refinement as follows: An orientable Engel structure induces a trivialization of the tangent bundle TQ via vector fields $\partial_{\mathcal{L}}$, $\partial_{\mathcal{D}}$, $\partial_{\mathcal{E}}$ and ∂_{TQ} where the first is tangent to the characteristic line field and the second lies inside the Engel structure such that $\partial_{\mathcal{L}}$ and $\partial_{\mathcal{D}}$ define a trivialization of the Engel structure (see [9, Theorem 3.37]).

Definition 4.1. The map $\phi_1: Q \rightarrow \xi_1$ which sends a point $q \in Q$ to the element $\phi_1(q) \in (\xi_1)_q$ associated to the subspace $T_q\pi(\partial_{\mathcal{D}}|_q)$ is called **oriented development map** of the Engel structure \mathcal{D} .

There is a canonical projection map $\pi_{\xi}: \xi_1 \rightarrow \mathbb{P}\xi$ and the triangle

$$\begin{array}{ccc} Q & \xrightarrow{\phi_{\mathcal{D}}} & \mathbb{P}\xi \\ \phi_1 \downarrow & \nearrow \pi_{\xi} & \\ \xi_1 & & \end{array}$$

obviously commutes. This implies that the twisting number n is even and that the equation $n/2 \cdot e(Q) = e(\xi_1) = e(\xi)$ holds (see Theorem 3.2).

Conversely, if the twisting number is even and the development map factorizes through the unit sphere bundle ξ_1 , then \mathcal{D} is orientable. Since this factorization is canonical upon a

choice of Riemannian metric on Q , an isotopy of oriented Engel structures corresponds to a homotopy of ϕ_1 through fiberwise covering maps.

Proof of Theorem 1.2. An Engel structure $\mathcal{D}^n(\xi)$ on Q with twisting number n and induced contact structure ξ exists if and only if $n \cdot e(Q) = e(\mathbb{P}\xi)$. Since $e(\mathbb{P}\xi)$ equals $2 \cdot e(\xi)$ we see that $\mathcal{D}^n(\xi)$ exists if and only if $n \cdot e(Q) = 2 \cdot e(\xi)$ which proves the first part of the theorem.

An orientable Engel structure on Q with twisting number n exists if and only if there is a fiberwise $n/2$ -fold covering map $Q \rightarrow \xi_1$. This is equivalent to the equality $n/2 \cdot e(Q) = e(\xi_1)$ (see Theorem 3.2). This proves the second part of the theorem.

To prove the third statement, observe that there is a simply transitive action of $H^1(M; \mathbb{Z})$ on $\text{Eng}_\xi^n(Q)$ defined by

$$\Phi: H^1(M; \mathbb{Z}) \times \text{Eng}_\xi^n(Q) \rightarrow \text{Eng}_\xi^n(Q), (\alpha, \phi_{\mathcal{D}}) \mapsto \alpha \cdot \phi_{\mathcal{D}}$$

where the product is defined as in (3.1), \mathcal{D} is an Engel structure on Q and $\phi_{\mathcal{D}}$ its development map (see Theorem 3.2 and Theorem 1.1). Here, we implicitly used the fact that Engel structures with characteristic line field tangent to the fibers and fiberwise covering maps can be considered as equivalent objects. The oriented Engel structures sit in $\text{Eng}_\xi^n(Q)$ as a subset. More precisely, the projection π_ξ induces a map

$$\iota_\xi: \text{Cov}_{\frac{n}{2}}(Q, \xi_1) \rightarrow \text{Eng}_\xi^n(Q), \phi \mapsto \pi_\xi \circ \phi$$

such that $\text{im}(\iota_\xi) = \text{Eng}_{\xi; o}^n(Q)$. Suppose that $\phi_{\mathcal{D}}$ factorizes through ξ_1 via ϕ_1 , then

$$(2\alpha) \cdot \phi_{\mathcal{D}} = (2\alpha) \cdot (\pi_\xi \circ \phi_1) = \pi_\xi \circ (\alpha \cdot \phi_1).$$

Therefore, the diagram

$$\begin{array}{ccccc} \alpha & & H^1(M; \mathbb{Z}) \xrightarrow{\cdot 2} H^1(M; \mathbb{Z}) & & 2 \cdot \alpha \\ \downarrow & & \cong \downarrow & & \downarrow \cong \\ \alpha \cdot \phi_1 & \xleftarrow{\iota_\xi} & \text{Cov}_{\frac{n}{2}}(Q, \xi_1) \xrightarrow{\iota_\xi} \text{Eng}_\xi^n(Q) & \xrightarrow{\pi_\xi \circ (\alpha \cdot \phi_1)} & (2\alpha) \cdot \pi_\xi \circ \phi_1 \end{array}$$

is commutative, which shows that Φ descends to a simply-transitive $2 \cdot H^1(M; \mathbb{Z})$ -action on $\text{im}(\iota_\xi) = \text{Eng}_{\xi; o}^n(Q)$, which finishes the proof. \square

Proof of Corollary 1.3. Since the Euler class $e(M \times \mathbb{S}^1)$ of the trivial circle bundle vanishes, the equation $n \cdot e(M \times \mathbb{S}^1) = 2 \cdot e(\xi)$ is fulfilled for every number n and contact structure ξ with Euler class/first chern class a 2-torsion class. So, Theorem 1.2 states that $\text{Eng}_\xi^n(M \times \mathbb{S}^1)$ bijects onto $H^1(M; \mathbb{Z})$ for all pairs $(n, \xi) \in \mathbb{Z} \times \Xi_2(M)$ which proves the first part of the corollary.

Suppose we are given an oriented Engel structure \mathcal{D} on $M \times \mathbb{S}^1$. Since \mathcal{D} is oriented it comes equipped with a trivialization. Since $M \times \mathbb{S}^1$ is a trivial circle bundle, by choosing a section of this circle bundle, the trivialization of \mathcal{D} induces a trivialization of the induced

contact structure ξ on the base space. So, the isotopy classes of oriented Engel structures can be thought of as a subset of $2\mathbb{Z} \times \Xi_0(M) \times H^1(M; \mathbb{Z})$. Conversely, every isotopy class of Engel structures inside the subset $2\mathbb{Z} \times \Xi_0(M) \times H^1(M; \mathbb{Z})$ fulfills the equation

$$n/2 \cdot e(M \times \mathbb{S}^1) = e(\xi)$$

and is therefore represented by an oriented Engel structure due to Theorem 1.2. \square

REFERENCES

- [1] G. E. BREDON, *Geometry and Topology*, Graduate Texts in Mathematics **139**, Springer-Verlag, 1993.
- [2] H. GEIGES, *An Introduction to Contact Topology*, Cambridge Studies in Advanced Mathematics **109**, Cambridge University Press, 2008.
- [3] A. GOLUBEV, *On the global stability of maximally nonholonomic two-plane fields in four dimensions*, *Int. Math. Res. Not.* **11** (1997), 523–529.
- [4] R. E. GOMPF AND A. I. STIPSICZ, *4-Manifolds and Kirby Calculus*, Graduate Studies in Mathematics **20**, American Mathematical Society, 1999.
- [5] A. HATCHER, *Algebraic Topology*, Cambridge University Press, 2002.
- [6] M. KLUKAS AND B. SAHAMIE, On prolongations of contact manifolds, *Proc. Amer. Math. Soc.* (to appear).
- [7] J. W. MILNOR, *Topology from the Differentiable Viewpoint*, Princeton Landmarks in Mathematics and Physics, Princeton University Press, 1997.
- [8] R. MONTGOMERY, Engel deformations and contact structures, *American Mathematical Society Translations* **196**(2), 103–117, American Mathematical Society, 1999.
- [9] T. VOGEL, Existence of Engel structures, PhD thesis, LMU München (2004).
- [10] T. VOGEL, Existence of Engel structures, *Ann. of Math.* **169** (2009), 79–137.

MATHEMATISCHES INSTITUT DER UNIVERSITÄT ZU KÖLN, WEYERTAL 86-90, 50931 KÖLN
 URL: <http://www.mi.uni-koeln.de/~mklukas>

MATHEMATISCHES INSTITUT DER LMU MÜNCHEN, THERESIENSTRASSE 39, 80333 MÜNCHEN GERMANY
 E-mail address: sahamie@math.lmu.de
 URL: <http://www.math.lmu.de/~sahamie>

CURRENT ADDRESS: STANFORD UNIVERSITY, 450 SLOAN HALL (BUILDING 380), STANFORD, CA-94043